PARAMETER ESTIMATION OF A WEIBULL DISTRIBUTION AND ACCELERATED LIFE TESTING

by

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1. INTRODUCTION

A Weibull distribution can often be used to model failure times of lifetime data. The form of the Weibull distribution that will be considered here is given as

$$\mathrm{pdf}(t;\ \alpha,\beta) = \begin{cases} \dfrac{\alpha}{\beta^{\alpha}} t^{\alpha-1} e^{-\left(\frac{t}{\beta}\right)^{\alpha}} & ,\ t \geq 0 \\ 0 & ,\ t < 0 \end{cases} \tag{1}$$

where α is the shape parameter and β is the characteristic life or the 63.2 percentile. The β parameter is commonly referred to as the scale parameter. The c.d.f. for the above form of the Weibull distribution is given as

$$\mathrm{cdf}(t;\ \alpha,\beta) = \begin{cases} 0 & ,\ t < 0 \\ 1 - e^{-\left(\frac{t}{\beta}\right)^{\alpha}} & ,\ t \geq 0 \end{cases} \tag{2}$$

An alternative form of this distribution given by Weibull (1951), is

$$\mathrm{cdf}(t) = \begin{cases} 1 - e^{-\left(\frac{t-\gamma}{\beta}\right)^{\alpha}} & ,\ t \geq \gamma \\ 0 & ,\ t < \gamma \end{cases}$$

where α and β are as before and γ is a location parameter. Note that (2) is just a special case of the form given by Weibull where the location parameter is taken to be zero. We will consider only the two parameter case which appears to be the most popular form of the distribution.

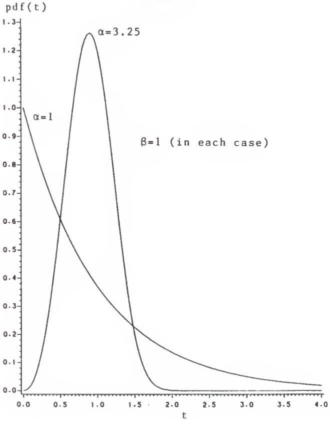
One advantage of using the Weibull distribution is that by varying α and β one can get a wide variety of distributional shapes. For instance, by letting $\alpha = 1$ we obtain an exponential distribution with mean β. When $\beta = 1$ and $\alpha \approx 3.25$, the Weibull distribution looks almost identical to the unit normal distribution as seen in Figure 1. The mean and variance for the Weibull distribution in general are

$$\mu = \beta\Gamma(1 + 1/\alpha)$$

and

$$\sigma^2 = \beta^2\{\Gamma(1 + 2/\alpha) - [\Gamma(1 + 1/\alpha)]^2\}.$$

FIGURE 1



One approach to estimating the parameters α and β is the graphical procedure. It is often used in practice to determine whether observed lifetime data can be adequately modeled by a Weibull distribution. This involves plotting the failure times on special Weibull reliability graph paper to see if a straight line adequately describes the data. If so, the data are assumed to be from a Weibull distribution, and the parameter estimates can be obtained from the graph. Another approach is maximum likelihood estimation. Solutions to the likelihood equations do not exist in closed form and need to be solved by numerical methods such as Newton-Raphson.

For the two approaches discussed in this paper, it is useful to make the logarithmic transformation. By doing this the Weibull distribution is transformed into an extreme value distribution which is derived below.

Let $Y = \ln(T)$. Then

$$\begin{aligned} P(Y \leq y) &= P(\ln(T) \leq y) \\ &= P(T \leq e^{y}) \\ &= 1 - e^{-\left(\frac{e^{y}}{\beta}\right)^{\alpha}}. \end{aligned}$$

Now let $\mu = \ln(\beta)$ and $\sigma = 1/\alpha$. Then

$$\begin{aligned} \mathrm{cdf}(y) &= 1 - e^{(e^{y}e^{-\mu})^{1/\sigma}} \\ &= 1 - e^{-e^{(y - \mu)/\sigma}}. \end{aligned}$$

The parameters μ and σ are location and scale parameters, respectively. The mean and variance of the extreme value distribution are

$$E(y) = \mu + 0.57722\sigma$$

and

$$\text{var}(y) = 1.64493\sigma^2$$

If $\mu = 0$ and $\sigma = 1$ we have the standard extreme value distribution where

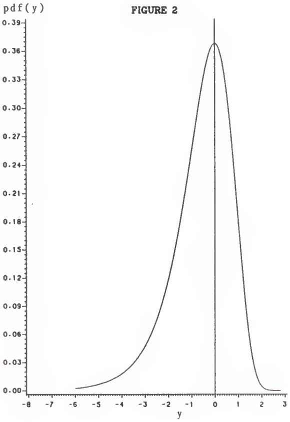
$$\text{cdf}(y) = 1 - e^{-e^{y}}$$

and

$$\text{pdf}(y) = e^{y}\, e^{-e^{y}}.$$

The shape of the standard extreme value density function is shown in Figure 2.

FIGURE 2



Now estimating μ and σ for the extreme value distribution provide estimates for α and β of the Weibull distribution. Section two will discuss a regression approach for estimating these parameters. In section three the best linear unbiased approach to estimating these parameters is given. Section four will compare these approaches. The last section will extend these ideas to accelerated life testing.

2. Regression Approach

Let F(y) denote the c.d.f. of the extreme value random variable Y. Recall,

$$F(y) = 1 - e^{-e^{(y-\mu)/\sigma}}$$

This implies that

$$\ln[-\ln(1 - F(Y))] = (Y - \mu)/\sigma .$$

Since the expression on the left is a linear function of Y we can observe a sample of n failure times $y_1, \ldots, y_n$ from an extreme value distribution and plot $\ln[-\ln(1 - F(y_i))]$ versus y_i to estimate μ and σ. Of course $F(y_i)$ is unknown, so we need to estimate this quantity to plot the data. The particular estimate used is commmonly referred to as the plotting position.

The first choice that will be looked at in this paper is the mean plotting position. First note that if Y is a random variable then $F(Y) \sim U(0,1)$. Now the observed failure times $y_1, \ldots, y_n$ are order statistics from an extreme value distribution so $F(y_i)$, $i = 1, \ldots, n$ will be order statistics from a Uniform (0,1) distribution. The expected value of the $i^{\underline{th}}$ order statistic from this distribution is $i/(n + 1)$ (Johnson and Kotz, 1970) and this is referred to as the mean plotting position.

Now we have the relationship

$$\ln[-\ln(1 - i/(n + 1))] \simeq (y_i - \mu)/\sigma .$$

Let $x_i = \ln[-\ln(1 - i/(n + 1))]$ for simplicity. By rewriting the linear relationship between x_i and y_i as

$$y_i \simeq \mu + \sigma x_i$$

the data can be plotted and the estimate of μ will be the intercept and the estimate of σ will be the slope.

Two other plotting positions that will be investigated in this paper are the midpoint plotting position and mean approximation plotting position. When the midpoint plotting position is used, $F(y_i)$ will be replaced by $(i - 0.5)/n$, and when using the mean approximation plotting position, $F(y_i)$ will be replaced by $(i - 0.375)/(n + 0.25)$.

Rather than estimating $F(y_i)$ one might consider estimating $-\ln(1 - F(y_i))$ which is the $i^{\underline{th}}$ order statistic from an exponential distribution. The expected value of this $i^{\underline{th}}$ order statistic is

$$\sum_{j=1}^{i} 1/(n-j+1) = 1/n + 1/(n-1) + 1/(n-2) + \ldots + 1/(n-i+1)$$

(Johnson and Kotz, 1970). In this case x_i would be equal to $\ln(\sum_{j=1}^{i} 1/(n-j+1))$ and the same procedure as before can be used.

Regardless of how x_i is chosen, the simple linear regression method of estimating μ and σ seem to be the appropriate thing to do next. The idea, of course, is to determine the values of $\hat{\mu}$ and $\hat{\sigma}$ that minimize $\Sigma(y_i - (\hat{\mu} + \hat{\sigma} x_i))^2$. Using the notation of the ordinary linear model, we let

$$\hat{\beta} = \begin{pmatrix} \hat{\mu} \\ \hat{\sigma} \end{pmatrix}.$$

The familiar OLS estimate of β is

$$\hat{\beta} = (X'X)^{-1} X'Y$$

where

$$X = \begin{pmatrix} 1 & x_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & x_n \end{pmatrix}$$

and Y is an n x 1 vector of order statistics from an extreme value distribution. It should be noted that weighted least squares should be used but the performance of this simpler estimator will be considered here.

Let Cov(Y) denote the covariance matrix of the random vector Y. It can be shown that Cov(Y) = σ^2 V where V is the covariance matrix from the standard extreme value distribution ($\mu = 0$ and $\sigma = 1$). Also analogous to the normal case note that if Y_i is a random variable from the extreme value distribution then $Z_i = (Y_i - \mu)/\sigma$ is a random variable from the standard extreme value distribution. Now to see that Cov(Y) = σ^2V first note that

$$\begin{aligned} Var(Y_i) &= Var(\mu + \sigma Z_i) \\ &= \sigma^2 \ Var(Z_i) \end{aligned}$$

and

$$\begin{aligned} Cov(y_i, y_j) &= E[(y_i - E(y_i))(y_j - E(y_j))] \\ &= E[((\mu + \sigma Z_i) - E(\mu + \sigma Z_i))((\mu + \sigma Z_j) - E(\mu + \sigma Z_j))] \\ &= E[(\sigma(Z_i - E(Z_i)))(\sigma(Z_j - E(Z_j)))] \\ &= \sigma^2 E[(Z_i - E(Z_i))(Z_j - E(Z_j))] \\ &= \sigma^2 Cov(Z_i, Z_j). \end{aligned}$$

So Cov(Y) is $\sigma^2 Cov(Z) = \sigma^2 V$. From this, the covariance matrix of $\hat{\beta}$ is obtained as follows:

$$Cov(\hat{\beta}) = (X'X)^{-1}X'(\sigma^2 V)[(X'X)^{-1}X']'$$
$$= \sigma^2 (X'X)^{-1} X'V X(X'X)^{-1}.$$

Tables of covariances for the standard extreme value distribution are not widely available but were published by (Mann, 1968). In her paper, tabled values were produced from a simulation study for sample sizes ranging from 1 to 25. In this paper the covariance matrices were simulated using Turbo Pascal on a Zenith microcomputer. The results were obtained by generating 1000 samples and compared closely to those obtained by Mann. The expected values and covariance values rarely differed from Mann's by more than 0.01 and 0.03 respectively.

This regression approach to estimating and applies to censored data as well as complete data. The types of censoring include right censoring, left censoring, and censoring which occurs when items are excluded from the sample due to breakage or any other reason which occurs randomly. The type of censoring considered here is right censoring or Type II censoring. This occurs when a number of items are placed on test and the process is observed until a fixed percentage, say, 60 or 80% have failed.

It should be noted here that the Y_i's are not independent because they are ordered observations. Thus, the usual regression theory as developed for independent observations does not apply. In particular, the estimates are biased, a fact which is discussed in section 4. The idea here is not to get the best estimates, necessarily, but to come up with estimates which are "good enough" to use in practice. A comparison of these estimates to the best linear unbiased estimates is given in section 4.

3. Best Linear Unbiased Approach

Let $Z_1, Z_2, \ldots, Z_n$ be the order statistics from the standard extreme value distribution with c.d.f.

$$F(Z) = 1 - e^{-e^{Z}}.$$

Note that $Z = (Y - \mu)/\sigma$ where Y is a random variable from the extreme value distribution. This gives us the relationship $Y = \mu + \sigma Z$. If we let $Y_1, Y_2, \ldots, Y_n$ be the order statistics from the extreme value distribution, we have the following set of equalities;

$$\begin{aligned} Y_1 &= \mu + \sigma Z_1 \\ Y_2 &= \mu + \sigma Z_2 \\ &\vdots \\ Y_n &= \mu + \sigma Z_n. \end{aligned}$$

In matrix notation,

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} 1 & Z_1 \\ \vdots & \vdots \\ 1 & Z_n \end{pmatrix} \begin{pmatrix} \mu \\ \sigma \end{pmatrix}.$$

Noting that

$$E[\underline{Y}] = \begin{pmatrix} 1 & E(Z_1) \\ \vdots & \vdots \\ 1 & E(Z_n) \end{pmatrix} \begin{pmatrix} \mu \\ \sigma \end{pmatrix},$$

we can see that it is in the form of a general linear model where $E[\underline{Y}] = X\beta$. Also, from the previous section we note again that $Cov(Y) = \sigma^2 V$ where V is known.

Now the Generalized Gauss Markov Theorem (Lloyd,1952) can be applied here to get the best linear unbiased estimates of μ and σ. This estimate is $\hat{\beta}$ where

$$\hat{\beta} = (X'V^{-1}X)^{-1} X'V^{-1}Y.$$

The covariance matrix of $\hat{\mu}$ and $\hat{\sigma}$ is given by

$$\mathrm{Cov}(\hat{\beta}) = \sigma^2(X'V^{-1}X)^{-1}.$$

This procedure can again be used when there is complete or censored data.

4. Comparison of Regression Estimates to BLUE's

The major advantage of using the regression approach for estimating μ and σ over the best linear unbiased approach is that we do not need to know V. The regression estimates are not expected to do as well as the BLUE's, but the aim of this section is to determine how favorably (or unfavorably) they compare in terms of bias and efficiency.

In order to obtain the bias of the regression estimates, it should be noted that the expected values of $\hat{\mu}$ and $\hat{\sigma}$ need only to be found in the case of sampling from the standard extreme value distribution. To see how this is accomplished, write the regression estimates of μ and σ as follows:

$$\hat{\sigma} = \frac{\Sigma(x_i - \overline{x})(y_i - \overline{y})}{\Sigma(x_i - \overline{x})^2}$$

$$= \frac{\Sigma(x_i - \overline{x})(\mu + \sigma Z_i - (\mu + \sigma\overline{Z}))}{\Sigma(x_i - \overline{x})^2}$$

$$= \frac{\sigma\Sigma(x_i - \overline{x})(Z_i - \overline{Z})}{\Sigma(x_i - \overline{x})^2}$$

Thus,

$$E(\hat{\sigma}) = \sigma E \frac{\Sigma(x_i - \overline{x})(Z_i - \overline{Z})}{\Sigma(x_i - \overline{x})^2}$$

$$= \sigma E(\hat{\sigma}_0)$$

where $\hat{\sigma}_0$ is the estimate of σ when sampling from the standard extreme value distribution. The bias is

$$E(\hat{\sigma} - \sigma) = E(\hat{\sigma}) - \sigma$$

$$= \sigma E(\hat{\sigma}_0) - \sigma$$

$$= \sigma[E(\hat{\sigma}_0) - 1]$$

where $E(\hat{\sigma}_0) - 1$ is the bias when sampling from the standard extreme value distribution.

The estimate for μ is written as

$$\hat{\mu} = \overline{y} - \hat{\sigma}\,\overline{x} = \mu + \sigma\,\overline{Z} - \hat{\sigma}\,\overline{x}$$

and

$$\begin{aligned} E(\hat{\mu}) &= \mu + \sigma E(\overline{Z}) - E(\hat{\sigma})\overline{x} \\ &= \mu + \sigma E(\overline{Z}) - \sigma E(\hat{\sigma}_0)\overline{x} \\ &= \mu + \sigma E(Z - \hat{\sigma}_0 x) \end{aligned}$$

where $\overline{Z} - \hat{\sigma}_0\overline{x} = \hat{\mu}_0$ is the estimate of μ when sampling from the standard extreme value distribution. The bias for estimating using the regression approach then is

$$\begin{aligned} E(\hat{\mu} - \mu) &= E(\hat{\mu}) - \mu \\ &= \sigma E(\overline{Z} - \hat{\sigma}_0\overline{x}) \\ &= \sigma E(\hat{\mu}_0). \end{aligned}$$

The efficiencies of the regression estimators will be obtained by finding MSE(BLUE)/MSE(Regression) where MSE(Regression) is Var(Regression) + $(\text{Bias})^2$. In Tables 1-4 the bias of the regression estimates and efficiencies are given for each of the four regression variations discussed. In each cell the top number is for $\hat{\mu}$ and the bottom is for $\hat{\sigma}$. All of the biases and efficiencies were determined for sample sizes of 10 and 20 and three amounts of censoring—0%, 20%, and 40%. The type of censoring that was looked at is right censoring or Type II censoring. Tables 1-3 are the results when using the mean, mean approximation, and midpoint plotting positions resp. and Table 4 was obtained when $-\ln(1 - F(y_i))$ was replaced by the expected value of the $i^{\underline{th}}$ order statistic from an exponential distribution.

When using the mean plotting position the bias for estimating μ is very small when there is no censoring. As censoring increases the estimate for μ becomes negatively biased. When there is no censoring, σ is overestimated, but the bias gets close to zero as the censoring increases to 40%. The efficiency is high for estimating μ with no censoring and decreases as censoring increases. Just the opposite occurs when estimating σ. The behavior exhibited in Table 4 (exponential mean plotting position) is almost identical except that the bias for estimating μ is even more negative for all levels of censoring. This larger (in absolute magnitude) bias thus causes the efficiencies for $\hat{\mu}$ to be lower than when using the mean plotting position for all combinations of sample size and censoring.

In Tables 2 and 3, the approximate mean and midpoint plotting positions appear to behave very much alike. The behavior of $\hat{\mu}$ in each case is almost identical to that shown in Table 1 (mean plotting position). The major difference is that in Tables 2 and 3 the bias for $\hat{\sigma}$ is very close to zero when there is no censoring as opposed to 40% censoring in Table 1. The bias increases as censoring increases so that $\hat{\sigma}$ is negatively biased at 40% censoring as much or more than $\hat{\sigma}$ is positively biased when there is no censoring in Table 1.

Even so, the efficiency is higher in all cases for $\hat{\sigma}$ in Tables 2 and 3 than in Table 1. About the only difference between Tables 2 and 3 is that the bias for $\hat{\sigma}$ is not as severe in Table 2. That is why the approximate mean plotting position appears to have the most to offer. Another possibility is to combine two of the procedures i.e. use one procedure for estimating μ and another for estimating σ. The best choice in that situation appears to be the mean plotting position for μ and either the approximate mean or midpoint plotting position for σ.

Table 1. Mean Plotting Position

Bias

Censoring

		0%	20%	40%
n = 10	μ	$.00^{+}\sigma$	$-.40\sigma$	$-.80\sigma$
	σ	$.17\sigma$	$.07\sigma$	$.03\sigma$
n = 20	μ	$-.02\sigma$	$-.40\sigma$	$-.80\sigma$
	σ	$.11\sigma$	$.01\sigma$	$-.03\sigma$

Efficiency

Censoring

		0%	20%	40%
n = 10	μ	.97	.45	.28
	σ	.49	.57	.71
n = 20	μ	.92	.26	.12
	σ	.45	.67	.73

Table 2. Approximate Mean Plotting Position

Bias

Censoring

		0%	20%	40%
n = 10	μ	$-.04\sigma$	$-.44\sigma$	$-.83\sigma$
	σ	$.02\sigma$	$-.08\sigma$	$-.14\sigma$
n = 20	μ	$-.00^{+}\sigma$	$-.42\sigma$	$-.82\sigma$
	σ	$.01\sigma$	$-.09\sigma$	$-.15\sigma$

Efficiency

Censoring

		0%	20%	40%
n = 10	μ	.94	.41	.26
	σ	.69	.76	.89
n = 20	μ	.91	.24	.12
	σ	.63	.71	.73

Table 3. Midpoint Plotting Position

Bias

Censoring

		0%	20%	40%
n = 10	μ	$-.05\sigma$	$-.45\sigma$	$-.85\sigma$
	σ	$-.04\sigma$	$-.14\sigma$	$-.21\sigma$
n = 20	μ	$-.01\sigma$	$-.43\sigma$	$-.83\sigma$
	σ	$-.03\sigma$	$-.13\sigma$	$-.19\sigma$

Efficiency

Censoring

		0%	20%	40%
n = 10	μ	.93	.39	.25
	σ	.76	.76	.88
n = 20	μ	.91	.23	.12
	σ	.66	.67	.68

Table 4. Exponential Mean Plotting Position

Bias

Censoring

		0%	20%	40%
n = 10	μ	−.12σ	−.53σ	−.94σ
	σ	.12σ	.02σ	−.03σ
n = 20	μ	−.05σ	−.47σ	−.88σ
	σ	.09σ	−.02σ	−.07σ

Efficiency

Censoring

		0%	20%	40%
n = 10	μ	.85	.32	.21
	σ	.52	.76	.80
n = 20	μ	.87	.20	.10
	σ	.50	.71	.77

5. An Improved Regression Procedure

The maximum likelihood estimate for estimating the mean of an exponential distribution when there is Type II censoring is $(t_1 + t_2 + \ldots + t_r + (n-r)t_r)/r$ where t_i is the $i^{\underline{th}}$ ordered observation. That is, if an investigator has decided to terminate a study after r of the n subjects have failed, the last (n-r) are set equal to the $r^{\underline{th}}$ one observed. If the same idea is applied to the regression approach considered here, the only additional thing that must change is that the last (n-r) rows of the X matrix are now equal to the $r^{\underline{th}}$ row.

The evaluation of the regression method using the mean plotting position was redone using the idea above. The results are given in Table 5. The bias for $\hat{\mu}$ was improved tremendously and the efficiencies increased generally to higher than .90. The bias that remained for $\hat{\sigma}$ was still not very satisfying, but at least it was consistent for varying amounts of censoring.

The approximate mean plotting position with this method results in a small and consistent bias for both $\hat{\mu}$ and $\hat{\sigma}$ over the different amounts of censoring studied. As shown in Table 6, the efficiencies for μ are high, and the efficiencies for σ are comparable to those of the best regression methods in Section 4.

Table 5. Improved Regression Procedure

Mean Plotting Position

Bias

Censoring

		0%	20%	40%
n = 10	μ	$.00^{+}\sigma$	$.00^{+}\sigma$	$.04\sigma$
	σ	$.17\sigma$	$.17\sigma$	$.20\sigma$
n = 20	μ	$-.02\sigma$	$.02\sigma$	$.05\sigma$
	σ	$.11\sigma$	$.11\sigma$	$.13\sigma$

Efficiency

Censoring

		0%	20%	40%
n = 10	μ	.97	.97	.90
	σ	.49	.49	.53
n = 20	μ	.92	.95	.87
	σ	.45	.54	.56

Table 6. Improved Regression Procedure

Appoximate Mean Plotting Position

Bias

Censoring

		0%	20%	40%
n = 10	μ	$-.04\,\sigma$	$-.03\,\sigma$	$-.03\,\sigma$
	σ	$.02\,\sigma$	$.02\,\sigma$	$.02\,\sigma$
n = 20	μ	$-.00^{+}\,\sigma$	$-.00^{+}\,\sigma$	$.00^{+}\,\sigma$
	σ	$.01\,\sigma$	$.01\,\sigma$	$.02\,\sigma$

Efficiency

Censoring

		0%	20%	40%
n = 10	μ	.94	.96	.95
	σ	.69	.74	.81
n = 20	μ	.91	.95	.91
	σ	.63	.74	.78

6. Accelerated Life Testing

In many situations we would like to estimate the parameters of a Weibull or some other distribution but the failure times are so long as it is not practical. For instance, suppose that a manufacturer of watch batteries develops a new battery that will last much longer than batteries currently being made. The manufacturer would like to say that the new batteries last an average of six years, which is two years longer than the batteries currently being made. The manufacturer would certainly not be able to wait for six years before marketing its new product, so it is this type of situation that calls for an accelerated life test.

In an accelerated life test the operating environment is changed in a way to speed up the failure times of the items under study. The variable or variables that are changed are commonly referred to as stress variables. The idea is to put the items on test at different levels of the stress variables. Using this information and standard regression techniques, extrapolation can be used to obtain estimates of the parameter of interest under normal operating conditions.

Model selection is probably the most important aspect of the accelerated life testing process. A model here is simply a function that describes the relationship between time and the stress variable. In the case of a single stress variable, call it S, the stress functions (or models) take on three common forms for Weibull time-to-failure T (Mann, Schafer and Singpurwalla, 1974). Let μ and σ denote the location and scale parameters of the extreme value distribution of $Y = \log(T)$. The first form is where is just a linear function of the stress variable. This model then takes the form

$$\mu = A + B * S.$$

Another is known as the <u>Arrhenius Model</u> where μ is a linear function of the reciprocal of S. That is,

$$\mu = A + B/S.$$

The last model is known as the <u>Power Rule Model</u> in which μ is a linear function of log(S). This is,

$$\mu = A + B * \log(S).$$

In all these models, σ is assumed to be constant for all S.

Now, for example, suppose we observe the data in Table 7 (Nelson, 1982), and we wish to estimate μ when the stress variable (kV) is at 25kV which is its normal state. It is known here that the distribution of failure times is Weibull. The estimates of μ and σ are given in Table 8 using the regression approach with approximate mean plotting position and taking into account the censored observations. The Arrhenius Model appears to be the most appropriate model for this data and the plot of $\hat{\mu}$ versus 1/kV as shown in Figure 3 shows that the relationship is linear.

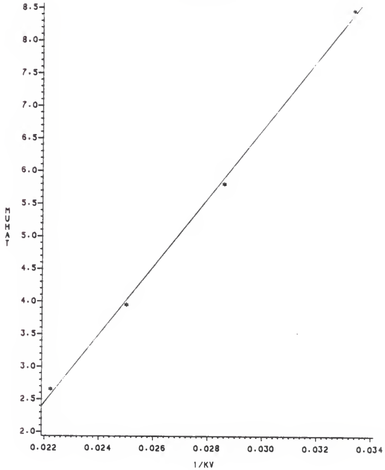
Table 7. Insulating Fluid Times (in seconds) to Breakdown with Censoring

45kV	40kV	35kV	30kV
1	1	30	50
1	1	33	134
1	2	41	187
2	3	87	882
2	12	93	1,448
3	25	98	1,468
9	46	116	2,290
13	56	258	2,932
47	68	461	4,138
50	109	1,182	15,750
55	323	1,350	$29{,}180^{+}$
71	417	1,495	$86{,}100^{+}$

Table 8

kV	$\hat{\mu}$	$\hat{\sigma}$
30	8.44	1.74
35	5.80	1.14
40	3.95	1.76
45	2.66	1.34

FIGURE 3



Using the method of least squares gives us A = -9.03 and B = 522.09 for this model and R^2 was .9987. Now we have the estimated stress function.

$$\hat{\mu}(kV) = -9.03 + 522.09/kV$$

and extrapolation to 25kV gives us 11.85 for an estimate of μ. We can assume the σ is constant over the different levels of kV since the data support the assumption here. An estimate of that value can be obtained by taking a simple or weighted average of $\hat{\sigma}$ for each level of kV. A simple average would seem more appropriate since the sample sizes are the same. In this case, a simple average gives us 1.495 for an estimate of μ. In terms of the original parameters of the Weibull distribution, $\hat{\beta}$ = 140,084 seconds and $\hat{\alpha}$ = 0.67 seconds at 25kV.

Now assuming the appropriate model has been chosen, confidence intervals for μ can be obtained. We can model the estimate of μ as

$$\hat{\mu} = A + B/kV + \varepsilon.$$

If we make the standard assumptions about ε for the simple linear regression model we can form a confidence interval for μ at kV = 25 as follows:

$$\hat{\mu} \pm t_{\alpha/2}\, s \sqrt{\frac{1}{n} + \frac{(x_p - \bar{x})^2}{SSxx}}$$

where $\hat{\mu} = -9.03 + 522.09/25 = 11.85$, $x_p = 1/25$, $\bar{x} = .027$ and $SS_{xx} = .00006909$ and $s = .112384$. So a 95% Confidence Interval for μ at 25kV is

$$11.85 \pm 3.182\ (.112384) \sqrt{\frac{1}{n} + \frac{(1/25\ -\ .027)^2}{.00006909}}$$

$$= 11.85 \pm .59 = (11.26,\ 12.44).$$

Note that this an approximate 95% Confidence Interval since the errors are only approximately normally distributed.

Now that the estimates for the parameters of the extreme value distribution (and hence the Weibull distribution) have been obtained, one can make inferences about the lifetimes of the items studied. This process could be done for competing product possibilities and then comparisons of estimated survival functions could be made. Also, it might be desirable to just choose the one with the highest median lifetime.

One must keep in mind when estimating μ beyond the range of the data that there is not only the usual sampling error, but there is error due to model selection. Different models should be compared to come up with the best one and the estimates should be examined using common sense. Also, using as many values of the stress variable as possible could only help.

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PARAMETER ESTIMATION OF A WEIBULL DISTRIBUTION
AND ACCELERATED LIFE TESTING

by

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ABSTRACT

The Weibull distribution can often be used to model situations in which failure rates of objects is of interest. Two methods of estimating the location and scale parameters of a Weibull distribution are examined. A regression approach to estimation is discussed and then compared the best linear unbiased method of estimation. An improved regression method is developed which results in a small bias. Accelerated life testing is used when it is not practical to observe the data because of long failure times. An example is given to illustrate the use of the regression approach in accelerated life testing.